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Derivation of Quantum Maxwell Equations from Relativistic Particle Hamiltonian

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Given the Hamiltonian for N relativistic particles with charges and intrinsic magnetic moments interacting via pair potentials and self-interactions, we derive not only the particle equations, but also the full set of Maxwell's equations, thereby testing the consistency of particle equations, currents, and field equations in the Heisenberg picture.

1. INTRODUCTION

In a recent formalism of quantum electrodynamics and radiative processes the electromagnetic field A_μ has been eliminated between the coupled Maxwell and Dirac equations in terms of the currents of the particles (sources) [for reviews see Barut (1988, 1991a)]. As a result one has a (nonlinear) Hamiltonian system of charged spinning particles interacting via pair potentials and via self-interactions. This Hamiltonian depends on the canonical variables of the particles alone. In a quantized theory these canonical variables are Heisenberg operators $\mathbf{x}$, $\mathbf{p}$ and Dirac spin operators $\boldsymbol{\alpha}$, $\boldsymbol{\sigma}$ of all the particles. Starting from such a Hamiltonian, we can derive not only the Heisenberg equations of the motion for the particles, but also, as we will show in this paper, the Maxwell equations for the fields $\mathbf{E}$ and $\mathbf{B}$ produced by all the particles. These fields will be operators because they depend on the canonical operators of the particles only. We have thus an operator field theory in terms of particles canonical operators. The currents j^μ also depend on the canonical variables of the particles. If the sources are quantized, so will be the fields produced by them. In this formulation

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E and **B** do not carry separate degrees of freedom; the quantization of the particles is enough. Quantized properties of fields (e.g., uncertainty relations) follow from those of the sources. There would be a double counting if we assigned independent degrees of freedom to particles and fields separately.

Because the original Hamiltonian itself has been constructed from the coupled Dirac and Maxwell equations, the reconstruction of Maxwell fields from the Hamiltonian is a consistency test of the equations of motion of the particles, their currents, and Maxwell's equations. Maxwell's equations are, of course, half of the physics. We must also know the currents and their equations, which are given by the equations of motion of the particles.

But by introducing fields into a particle theory we can derive a certain duality principle that physical quantities (e.g., energy-momentum tensor) can be thought to reside either in the field or in the particles.

When we consider a subset of the particles as our system, say a single particle, and when the dynamics of other particles is of no interest to us, then this dynamics can be put into the field, which then acts as an external dynamical field to our system.

Special attention must be given to the self-field of the particles, because the consistency of the formalism requires the inclusion of self-fields, which is the cause of radiative processes in quantum electrodynamics, such as the Lamb shift, the anomalous magnetic moment, and spontaneous emission.

2. HAMILTONIAN AND PARTICLE EQUATIONS

We start from N Dirac particles having charges e_i (and for completeness anomalous magnetic moments a_i) interacting via electric and magnetic pair potentials. The Hamiltonian is (Barut and Xu, 1982; Barut, 1991*b*)

$$H = \sum_{i=1}^{N} \left\{ \boldsymbol{\alpha}_i \cdot \left[\mathbf{p}_i - e_i \sum_{j=1}^{N} \mathbf{A}_j(\mathbf{x}_i) \right] + \beta_i m_i + e_i \sum_j \phi_j(\mathbf{x}_i) - a_i \left[(\beta \boldsymbol{\sigma}_i) \cdot \sum_{j=1}^{N} \mathbf{B}_{aj}(\mathbf{x}_i) + i(\beta \boldsymbol{\alpha}_i) \cdot \sum_{j=1}^{N} \mathbf{E}_{aj}(\mathbf{x}_i) \right] \right\} \tag{1}$$

where the two-body potentials are

$$\phi_j(\mathbf{x}_i) = \frac{e_j}{4\pi |\mathbf{x}_i - \mathbf{x}_j|} - i \frac{a_j}{\pi} \frac{(\mathbf{x}_i - \mathbf{x}_j) \cdot (\beta \boldsymbol{\alpha})_j}{|\mathbf{x}_i - \mathbf{x}_j|^3} \tag{2}$$

$$\mathbf{A}_j(\mathbf{x}_i) = \frac{e_j \boldsymbol{\alpha}_j}{4\pi |\mathbf{x}_i - \mathbf{x}_j|} + \frac{a_j}{\pi} (\beta \boldsymbol{\sigma})_j \times \frac{(\mathbf{x}_i - \mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^3} \tag{3}$$

The electric and magnetic anomalous fields $\mathbf{E}_a$ and $\mathbf{B}_a$ entering in (1) are obtained from the magnetic parts of the potentials in (2) and (3):

$$\mathbf{E}_{aj}(\mathbf{x}_i) = -i\frac{a_j}{\pi}\left[\frac{3(\mathbf{x}_i-\mathbf{x}_j)(\beta\boldsymbol{\alpha})_j\cdot(\mathbf{x}_i-\mathbf{x}_j)}{|\mathbf{x}_i-\mathbf{x}_j|^5}-\frac{\beta\boldsymbol{\alpha}_j}{|\mathbf{x}_i-\mathbf{x}_j|^3}-\frac{4\pi}{3}(\beta\boldsymbol{\alpha})_j\delta(\mathbf{x}_i-\mathbf{x}_j)\right] \tag{4}$$

$$\mathbf{B}_a(\mathbf{x}_i) = \frac{a_j}{\pi}\left[\frac{3(\mathbf{x}_i-\mathbf{x}_j)(\beta\boldsymbol{\sigma})_j\cdot(\mathbf{x}_i-\mathbf{x}_j)}{|\mathbf{x}_i-\mathbf{x}_j|^5}-\frac{\beta\boldsymbol{\sigma}_j}{|\mathbf{x}_i-\mathbf{x}_j|^3}+\frac{8\pi}{3}(\beta\boldsymbol{\sigma})_i\delta(\mathbf{x}_i-\mathbf{x}_j)\right] \tag{5}$$

We derive first from the Hamiltonian H, equation (1), the Heisenberg equations of motion for the particles

$$\dot{\mathbf{x}}_i = i[H, \mathbf{x}_i] = \boldsymbol{\alpha}_i \tag{6}$$

$$\dot{\mathbf{p}}_i = i[H, \mathbf{p}_i] = -e\nabla_i \sum_j \phi_j(\mathbf{x}_i) + e_i \sum_j \nabla_i(\boldsymbol{\alpha}_i\cdot\mathbf{A}_j(\mathbf{x}_i)) + a_i\nabla_i\sum_j[(\beta\boldsymbol{\sigma})_i\cdot\mathbf{B}_{aj} - i(\beta\boldsymbol{\alpha}_i)\cdot E_{aj}] \tag{7}$$

The kinetic momenta are

$$\pi_i = \mathbf{p}_i - \mathbf{e}_i\sum_j \mathbf{A}_j(\mathbf{x}_i) \tag{8}$$

so that

$$\dot{\pi}_i = \dot{\mathbf{p}}_i - \mathbf{e}_i\sum_j \dot{\mathbf{A}}_j(\mathbf{x}_i) \tag{9}$$

The last term itself can be evaluated from the Heisenberg equation

$$\dot{\mathbf{A}}_j(\mathbf{x}_i) = i[H, A_j(\mathbf{x}_i)] = (\boldsymbol{\alpha}_i\cdot\nabla_i)\mathbf{A}_j(\mathbf{x}_i) \tag{10}$$

so that $\dot{\pi}_i$ is given by (9) together with (7) and (10), which we can write as

$$\dot{\pi}_i = e_i(\mathbf{E}+\dot{\mathbf{x}}_i\wedge\mathbf{B}) + a_i[(\beta\sigma)_i\cdot\nabla\mathbf{B}_a - i(\beta\sigma)_i\cdot\nabla\mathbf{E}_a] \tag{11}$$

where we have defined

$$\begin{aligned}\mathbf{E}(\mathbf{x}) &= -\nabla_x\sum_j \phi_j(\mathbf{x})\\ \mathbf{B}(\mathbf{x}) &= \nabla_x\times\sum_j \mathbf{A}_j(\mathbf{x})\\ \mathbf{E}_a(\mathbf{x}) &= \sum_j \mathbf{E}_{aj}\\ \mathbf{B}_a(\mathbf{x}) &= \sum_j \mathbf{B}_{aj}\end{aligned} \tag{12}$$

where **x** is the position of the particle *i.* Note that the interparticle potentials have no explicit time dependence, and that in the sums over j in equation (12) the self-fields are also included.

We see in equation (11) the Lorentz force (including the self-force, which must be renormalized) as well as the forces on the dipole depending on the gradient of the fields. Equation (11) can also be obtained directly from $\dot{\pi} = i[H, \pi)$. To complete the dynamical system of equations for the particle variables $\{\mathbf{x}, \boldsymbol{\pi}, \boldsymbol{\alpha}, \boldsymbol{\sigma}\}$, we evaluate the remaining Heisenberg equations

$$\begin{aligned}\dot{\boldsymbol{\alpha}}_i &= 2\pi_i \times \sigma_i - 2\gamma_i m_i + 2a_i[\gamma_i(\sigma_i \cdot \mathbf{B}_a) - \mathbf{B}_a \wedge (\beta\boldsymbol{\alpha})_i - \beta\mathbf{E}_a] \\ &\quad -2\sum_j \frac{e_j}{4\pi}\frac{1}{|\mathbf{x}_i - \mathbf{x}_j|}\boldsymbol{\alpha}_j \wedge \boldsymbol{\sigma}_i \end{aligned} \qquad (13)$$

$$\begin{aligned}\dot{\boldsymbol{\sigma}}_i &= 2\pi_i \times \boldsymbol{\alpha}_i - 2a_i[\mathbf{B}_a \wedge (\beta\boldsymbol{\sigma})_i - \mathbf{E}_a \wedge (i\beta\boldsymbol{\alpha})_i] \\ &\quad -2\sum_j \frac{e_j}{4\pi}\frac{1}{|\mathbf{x}_i - \mathbf{x}_j|}\boldsymbol{\alpha}_j \times \boldsymbol{\alpha}_i \end{aligned} \qquad (14)$$

Equations (6), (11), (13), and (14) define our dynamical system completely from the point of view of particle variables. In the next section we study the field equations.

3. THE OPERATOR MAXWELL EQUATIONS

From the definitions in (12) we have immediately

$$\nabla \cdot \mathbf{B} = 0 \qquad (15)$$

and

$$\begin{aligned}\nabla \cdot \mathbf{E} &= -\nabla^2 \sum_j \phi_j = j_0(\mathbf{x}) \\ j_0(\mathbf{x}) &= \sum_j [e_j\delta(\mathbf{x} - \mathbf{x}_j) + i4a_j(\beta\boldsymbol{\alpha})_j \cdot \nabla\delta(x - x_j)]\end{aligned} \qquad (16)$$

where we have used the relations ($\mathbf{a}$ = fixed vector)

$$\nabla \cdot \left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) = 0, \qquad \nabla \cdot \nabla\left(\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}\right) = 4\pi(\mathbf{a}\cdot\nabla)\delta(\dot{\mathbf{r}}) \qquad (17)$$

Next we evaluate the remaining two Maxwell equations. Becuase we have in our system no explicit time dependence (see Section 4), we have

$$\partial E/\partial t = 0, \qquad \partial B/\partial t = 0 \qquad (18)$$

Consequently, the third Maxwell equation is automatically satisfied,

$$\partial B/\partial t = -\nabla \times E = -\nabla \times (\nabla \phi) \tag{19}$$

as both sides are automatically zero.

Finally, the last Maxwell equation, with (18), determines the current **j**, by

$$\mathbf{j} = \nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$\mathbf{j}(\mathbf{x}) = \sum_j e_j \boldsymbol{\alpha}_j \delta(\mathbf{x}-\mathbf{x}_j) - 4a_j(\beta\boldsymbol{\sigma})_j \times \nabla\delta(\mathbf{x}-\mathbf{x}_j) \tag{20}$$

Here we have used the following identities:

$$\begin{aligned} \nabla \times \left(\mathbf{m} \times \frac{\mathbf{r}}{r^3}\right) &= 3(\mathbf{m}\cdot\mathbf{r})\frac{\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3} + \frac{8\pi}{3}\mathbf{m}\delta(\mathbf{r}) \\ -\nabla\left(\mathbf{m}\cdot\frac{\mathbf{r}}{r^3}\right) &= 3(\mathbf{m}\cdot\mathbf{r})\frac{\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3} - \frac{4\pi}{3}\mathbf{m}\delta(\mathbf{r}) \end{aligned} \tag{21}$$

so that, using (3) and (21),

$$\begin{aligned} \nabla \times \nabla \times \left(\frac{e}{4\pi}\frac{\boldsymbol{\alpha}}{r}\right) &= \nabla \times \left(\frac{e}{4\pi}\boldsymbol{\alpha} \times \frac{\mathbf{r}}{r^3}\right) \\ &= \frac{e}{4\pi}\left[-\nabla\left(\boldsymbol{\alpha}\cdot\frac{\mathbf{r}}{r^3}\right) + 4\pi\boldsymbol{\alpha}\delta(\mathbf{r})\right] \end{aligned} \tag{22}$$

The contribution of the first term vanishes if we use the current conservation. For

$$\begin{aligned} \sum_j \frac{e_j}{4\pi}\nabla\left(\boldsymbol{\alpha}_j \cdot \frac{\mathbf{r}}{r^3}\right) &\rightarrow \frac{1}{4\pi}\int d\mathbf{x}'\, e_j\boldsymbol{\alpha}_j \cdot \nabla' \frac{1}{|\mathbf{x}-\mathbf{x}'|} \\ &= -\frac{1}{4\pi}\int d\mathbf{x}'\, \nabla \cdot (e_j\boldsymbol{\alpha}_j)\frac{1}{|x-\chi'|} = 0 \end{aligned}$$

where we have integrated by part and used $\nabla \cdot (e_j\alpha_j) = \nabla \cdot j = 0$.

Note that from (16) and (20) the relativistic nature of the current $(j_0, \mathbf{j})$ can be seen.

4. EXTENSIONS OF THE FORMALISM

Several extensions of the previous formulas are possible.

(a) If we had in the Hamiltonian (1) an additional time-dependent field whose sources are far away or of not dynamical interest so that it can be treated as a C-number field $A_{\text{ext}}(\mathbf{x}, t)$, then the modifications are as follows. In the definition of the fields, equation (12), we get the additional terms

$$\mathbf{E}=-\nabla\phi-\frac{\partial}{\partial t}\mathbf{A}^{\text{ext}}$$
$$\mathbf{B}=\nabla\times\mathbf{A}+\nabla\times\mathbf{A}^{\text{ext}} \tag{23}$$

so that the Lorentz equations (11) remain unchanged with the new fields (23). The two Maxwell's equations (15)-(16) remain unchanged. The third Maxwell equation (19) becomes

$$\frac{\partial}{\partial t}(\nabla\times A^{\text{ext}})=-\nabla\times E=-\nabla\times\left(-\nabla\phi-\frac{\partial}{\partial t}A^{\text{ext}}\right) \tag{24}$$

which is again clearly satisfied.

In the last Maxwell equation, equation (20), we have

$$\mathbf{j}=\nabla\times(\nabla\times\mathbf{A}+\nabla\times\mathbf{A}^{\text{ext}})+\frac{\partial^2\mathbf{A}^{\text{ext}}}{\partial t^2}$$
$$=\nabla\times(\nabla\times A)+\nabla(\nabla\cdot\mathbf{A}^{\text{ext}})+\Box\mathbf{A}^{\text{ext}} \tag{25}$$

If $\mathbf{A}^{\text{ext}}$ is a free field (whose sources are at infinity), we have $\Box\mathbf{A}^{\text{ext}}=0$, and if $\nabla\cdot\mathbf{A}^{\text{ext}}=0$, then the current in (20) is unchanged.

(b) We may use a fully covariant formalism. Consider the mass operator $\mathscr{H}$ of the number of Dirac particles

$$\mathscr{H}=\sum_j\{\gamma_j^\mu(p_\mu^j-e_jA_\mu(x_j))-m_j+a_j\sigma_{\mu\nu}^jF^{\mu\nu}(x_j)\}$$

We then have

$$\dot{x}_\mu^j=i[\mathscr{H},x_\mu^j]=\gamma_\mu^j$$
$$\gamma_\mu^j=i[\mathscr{H},\gamma_\mu^j]=(p_\nu^j-eA_\nu^j)[\gamma^{\nu j},\gamma_\mu^j]+a_jF^{\nu\lambda}[\sigma_{\nu\lambda}^j,\gamma_\mu^j]$$
$$\dot{\pi}_\mu=i[\mathscr{H},p_\mu-eA_p]=eF_{\mu\nu}\gamma^\nu+a\sigma_{\lambda\nu}F^{\lambda\nu}_{\ ,\mu}$$

where $F=dA$. The two Maxwell equations follow from this, $\operatorname{div}B=0$, $\partial B/\partial t=-\nabla\times E$. So far we have not used the explicit form of A_μ.

If we take for A_μ the sum of the Lienard–Wiechert potentials, then we can derive the explicit form of the currents which is present in the Lienard–Wiechert potentials:

$$\nabla\cdot E=\rho,\qquad -\partial E/\partial t+\nabla\times B=\mathbf{j}$$

This is clearly to be expected because the Lienard–Wiechert potentials are obtained from the current.

(c) The preceding calculations can also be extended to the case where we have magnetic monopole and electric dipole currents, for which the two-body potentials are also known (Barut and Xu, 1983).

5. CONCLUSIONS

The Maxwell–Dirac action of electrodynamics (and its extension to include the anomalous magnetic moment coupling) leads by elimination of the electromagnetic potentials to an N-body Hamiltonian of interacting particles including self-interactions. From such a Hamiltonian we have reconstructed in this work the Maxwell equations in which the field **E** and **B** operators are determined by the dynamical variables of the source—the particles. The self-consistency of the two procedures shows that the electromagnetic field need not be quantized again independently. It is sufficient to quantize the particles, which is already done by the *starting* Dirac equations. The quantized properties of the fields follow from these, as either the sources or the detectors. The radiative processes are fully contained in the self-energy terms.

The problem of deriving Maxwell's equations from dynamics in a simple case was discussed recently by Dyson, who attributed it to Feynman (Dyson, 1990; see also Dombey *et al.*, 1991). There is also a similar discussion by Tu *et al.* (1978). We tried to show in this work the full scope of the relationship between particle dynamics and Maxwell's equations.

Finally note that the Breit-type magnetic interactions are included in equation (1): when (2) and (3) are inserted into (1) we see terms of the form $(1+\alpha_i\cdot\alpha_j)/|\mathbf{x}_i-\mathbf{x}_j|$. Equation (1) is essentially the Hamiltonian of a covariant one-time N-body equation. The time here is an invariant center-of-mass time, not proper time, and hence retardation problems do not arise (Barut, 1991*b*). We have for simplicity omitted the nonlinear self-energy radiative terms, which can be found in Barut (1991*b*). These do not change the general form of our results. Note further that both charge and currents, equations (16) and (20), also contain magnetic terms which usually do not figure in the models of charged particle dynamics.

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